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Second semester
Answer type "Analysis 2"

ercice 1. (5 pts) We have

a. (2 pts) To calculate $\int_{\frac{\pi}{2}}^{\pi} (\sin x)^{2024} \cos x dx$, we apply the Variable Change Method $u = \sin x$, which gives $du = \cos x dx$. So

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\pi} (\sin x)^{2024} \cos x dx &= \int_{\sin\left(\frac{\pi}{2}\right)}^{\sin(\pi)} u^{2024} du = \int_{1}^{0} u^{2024} du = \\ -\int_{0}^{1} u^{2024} du &= -\frac{u^{2025}}{2025}]_0^1 = -\frac{1}{2025}.\end{aligned}$$

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b. (3 pts) To calculate $\int_{-1}^{1} x \arctan x dx$, we apply the method of integration by parts, we set $u' = x$ and $v = \arctan x$.
This gives $u = \frac{1}{2}x^2$ and $v' = \frac{1}{x^2+1}$. Then

$$\begin{aligned}\int_{-1}^{1} x \arctan x dx &= \frac{1}{2}x^2 \arctan x]_{-1}^{1} - \int_{-1}^{1} \frac{1}{2}x^2 \left(\frac{1}{x^2+1}\right) dx \\ &= \frac{1}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] - \frac{1}{2}\int_{-1}^{1} \frac{x^2}{x^2+1} dx \\ &= \frac{\pi}{4} - \frac{1}{2}\int_{-1}^{1} \frac{x^2+1-1}{x^2+1} dx = \frac{\pi}{4} - \frac{1}{2}\left[\int_{-1}^{1} \left(1 - \frac{1}{x^2+1}\right) dx\right] \\ &= \frac{\pi}{4} - \frac{1}{2}[x - \arctan x]_{-1}^{1} = \frac{\pi}{4} - \frac{1}{2}\left[2 - \left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)\right] \\ &= \frac{\pi}{2} - 1.\end{aligned}$$

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ercice 2. (5 pts) Let the function defined on the interval $[1,5]$ by $f(x) = 3x^2 + 1$.

a. (1 pt) To show by recurrence that

$$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}, \quad (1)$$

we put

$$S_n = 1^2 + 2^2 + ... + n^2 \text{ and } \pi_n = \frac{n(n+1)(2n+1)}{6}.$$

The first property

$$S_1 = 1^2 = \frac{1.(1+1).(2.1+1)}{6} = \pi_1.$$

is satisfied.
Suppose (1) is satisfied until the order n and we're going to prove it for the order $n+1$. We have

$$\begin{aligned} S_{n+1} &= S_n + (n+1)^2 = \frac{n(n+1)(2n+1}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} = \frac{(n+1)}{6}[n(2n+1) + 6(n+1)] \\ &= \frac{(n+1)[2n^2 + 7n + 6]}{6} = \frac{(n+1)(n+2)(2n+3)}{6}. \end{aligned}$$

Then S_{n+1} can be written on the form (1):

$$S_{n+1} = \frac{(n+1)[(n+1)+1][2(n+1)+]}{6}.$$

This ends the proof of the part (a).

b. (4 pts) To calculate the integral

$$\int_1^5 f(x)dx,$$

we use the Riemann sums, we have by definition

$$\int_1^5 f(x)dx = \lim_{n\to\infty} P_n,$$

where

$$P_n = h_n \sum_{i=0}^{n} f(1 + ih_n),$$

with

$$h_n = \frac{(5-1)}{n}.$$

So

$$\begin{aligned} P_n &= h_n \sum_{i=0}^{n} \left[3\left(1 + \frac{4i}{n}\right)^2 + 1\right] = \frac{4}{n}\sum_{i=0}^{n}\left[3\left(1 + \frac{8i}{n} + \frac{16i^2}{n^2}\right) + 1\right] \\ &= \frac{4}{n}\left[\sum_{i=0}^{n} 3 + 1 + \frac{24}{n}\sum_{i=0}^{n} i + \frac{48}{n^2}\sum_{i=0}^{n} i^2\right]. \end{aligned}$$

Applying (1) without forgetting the well known formula

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2},$$

we get

$$\begin{aligned} P_n &= \frac{4}{n}\left[4n + \frac{24}{n}\frac{n(n+1)}{2} + \frac{48}{n^2}\frac{n(n+1)(2n+1)}{6}\right] \\ &= 4\left[4 + \frac{12(n+1)}{n}\frac{48(n+1)(2n+1)}{6n^2}\right] = 4[4+12+16] \\ &= 128. \end{aligned}$$

Which gives

$$\lim_{n\to\infty} P_n = 128.$$

Finally we have

$$\int_1^5 \left(3x^2+1\right) dx = 128.$$

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ercice 3. (5 pts) Let be the following ordinary differential equation:

$$x\left(x^2+1\right)\frac{dy}{dx} - 2y = x^3(x-1)e^{-x}. \tag{2}$$

a. (1 pt) We have

$$\frac{2}{x(x^2+1)} = \frac{2}{x} - \frac{2x}{x^2+1}$$

b. (1 pt) The homogeneous equation can be written as follows

$$x\left(x^2+1\right)\frac{dy}{dx} - 2y = 0,$$

which gives

$$\int \frac{dy}{y} = \int \frac{2}{x(x^2+1)}dx = \int \left(\frac{2}{x} - \frac{2x}{x^2+1}\right) dx.$$

Then

$$\ln\frac{y}{C} = 2\ln x - \ln\left(x^2+1\right),$$

where C is an arbitrary constant. Then

$$y = C\frac{x^2}{x^2+1}.$$

c. (2 pts) The method of changing the constant consists in assuming

$$y = C(x)\,\frac{x^2}{x^2+1},$$

then we replace in (2) to obtain

$$C'(x) = (x-1)\,e^{-x}.$$

Integration by parts gives

$$C(x) = -xe^{-x} + C_1$$

where C is an arbitrary constant. Then the general solution of the differential equation (2) can be written as follows

$$y = \frac{\left(-xe^{-x} + C_1\right)x^2}{x^2+1}.$$

d. (1 pt) For $x = 1$, we have $y(1) = \frac{-e^{-1}+C_1}{2} = 0$. Which gives $C_1 = e^{-1}$. The solution of the differential equation (2) satisfying the initial condition $y(1) = 0$ is

$$y = \frac{\left(-xe^{-x} + e^{-1}\right)x^2}{\left(x^2+1\right)}.$$

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ercice 4. (5 pts) Let the differential equation of the second order

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 27\left(2x^2+3x-1\right). \tag{3}$$

a. (2 pts) To find the general y_G solution of the homogeneous differential equation associated with (3), we solve the characteristic equation

$$\alpha^2 - 6\alpha + 9 = 0,$$

which gives $\alpha_1 = \alpha_2 = 3$, then

$$y_G = (C_1 + C_2x)\,e^{3x}.$$

b. (2 pts) To Find a Particular Solution of the Differential Equation (3) on the form $y_P = ax^2 + bx + c$, we replace in (3) and by identification we obtain

$$y_P = 6x^2 + 17x + 7.$$

c. (2 pts) The general solution of the differential equation (3) is

$$y = y_G + y_P = (C_1 + C_2x)\,e^{3x} + 6x^2 + 17x + 7.$$