Abbes Laghrour University of Khenchela | **University year:** 2023/2024
Department of Mathematics and computer science | **First year M/I/MI**
Duration: 1.5 Hour

ALGEBRA 2 FINAL EXAM CORRECTION

Exercise 01: (04 points)
I) Let $P_3(\mathbb{R})$ be the vector space of polynomials with real coefficients and degree at most 3. Let

$$W = \{p \in P_3(\mathbb{R}) : p(0) = p''(0) \text{ and } p'(1) = 0\},$$

where p' and p'' are the first and the second derivative of p respectively.

1• **Prove that W is a subspace of $P_3(\mathbb{R})$.**

Denote $p_0(x) = 0_{P_3(\mathbb{R})}$. Then $p_0(x) = 0$, $\forall x \in \mathbb{R}$.
We have $p_0(0) = p_0''(0) = 0$, and $p_0'(1) = 0$. Hence $p_0 \in W$, thus $W \neq \phi$. **(0.25)**

Let $p_1, \quad p_2 \in W$, then we have

$$\begin{cases} p_1(0) = p_1''(0) \text{ and } p_1'(1) = 0 \\ p_2(0) = p_2''(0) \text{ and } p_2'(1) = 0, \end{cases}$$

thus

$$\begin{cases} p_1(0) + p_2(0) = p_1''(0) + p_2''(0) \\ \text{and} \\ p_1'(1) + p_2'(1) = 0 \end{cases}$$

hence $p_1+p_2 \in W$. **(1.50)**
Let $\alpha \in \mathbb{R}$, let $p \in W$, we have

$$p(0) = p''(0) \text{ and } p'(1) = 0,$$

thus

$$\alpha p(0) = \alpha p''(0) \text{ and } \alpha p'(1) = \alpha 0 = 0.$$

hence $\alpha p \in W$.
thus W is a subspace of $P_3(\mathbb{R})$.

2• **Find a basis of W. Deduce the dimension of W.**
let $p \in W$ such that $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, with $a_i \in \mathbb{R}, 0 \leq i \leq 3$, then by the condition $p(0) = p''(0)$ and $p'(1) = 0$, we get $a_0 = 2a_2$ and $a_1 = -2a_2 - 3a_3$ respectively.
then we obtain **(02)**

$$\begin{aligned} p(x) &= 2a_2 + (-2a_2 - 3a_3)x + a_2x^2 + a_3x^3 \\ &= (2 - 2x + x^2)a_2 + (-3x + x^3)a_3, \end{aligned}$$

hence $W = \langle 2 - 2x + x^2, -3x + x^3 \rangle$, also we have $\{2 - 2x + x^2, -3x + x^3\}$ is linearly independent, thus the set $\{2 - 2x + x^2, -3x + x^3\}$ is a basis of W.

Consequently $\dim W = 2$. **(0.25)**

Exercise 02: (03 points)
Let H_1, H_2 and H_3 be three subspaces of $\mathbb{R}^3$ defined by

$$H_1 = \{(x, y, 0)\,;x, y \in \mathbb{R}\}, \quad H_2 = \{(0, y, z)\,;y, z \in \mathbb{R}\},$$
$$H_3 = \{(0, 0, z)\,;z \in \mathbb{R}\}.$$

1• **Are H_1 and H_2 a direct sum of $\mathbb{R}^3$? Justify your answer.**

No they are not because $H_1 \cap H_2 = (0, 1, 0) \neq \{0_{\mathbb{R}^3}\}$. **(01)**

2• **Prove that $\mathbb{R}^3 = H_1 \oplus H_3$.**

Let $(x, y, z) \in \mathbb{R}^3$, we have $(x, y, z) = (x, y, 0) + (0, 0, z)$, thus $(x, y, z) \in H_1 + H_3$,
hence $\mathbb{R}^3 \subset H_1 + H_3$,
since $H_1 \subset \mathbb{R}^3$ and $H_1 \subset \mathbb{R}^3$ then $H_1+H_3 \subset \mathbb{R}^3$, , for that we obtain $\mathbb{R}^3 = H_1+H_3$. **(01)**

Let $(x, y, z) \in H_1 \cap H_3$, then $\left\{ \begin{array}{c} (x, y, z) \in H_1 \\ \wedge \\ (x, y, z) \in H_3 \end{array} \right. \Rightarrow \left\{ \begin{array}{c} z = 0 \\ \wedge \\ x = y = 0 \end{array} \right. \Rightarrow (x, y, z) = (0, 0, 0)$, **(01)**
then $H_1 \cap H_3 = \{0_{\mathbb{R}^3}\}$.Thus $\mathbb{R}^3 = H_1 \oplus H_3$.

Exercise 03: (04 points)
Let A be a matrix defined as:

$$A = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{pmatrix}.$$

1• **Calculate $(A - 2I)^3$, then deduce that A is invertible.**

$(A - 2I)^3 = (A - 2I)^2(A - 2I)$,

We have $(A-2I) = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix}$ **(0.50)**

first we calculate $(A - 2I)^2$.

$(A-2I)^2 = (A-2I)(A-2I) = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ **(01)**

Hence $(A-2I)^3 = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ **(01)**

we have $(A - 2I)^3 = A^3 - 6A^2 + 12A - 8I = 0_{\mathcal{M}_3(\mathbb{R})}$

so we obtain $A(\frac{1}{8}A^2 - \frac{3}{4}A + \frac{3}{2}I) = I$. **(01)**

hence A is invertible.

2• **Define A^{-1} in terms of I, A and A^2.**
$A^{-1} = \frac{1}{8}A^2 - \frac{3}{4}A + \frac{3}{2}I$. **(0.50)**

Exercise 04: (09 points)
I)Let f be a map defined as:

$$f: \begin{array}{ccc} \mathbb{R}^3 & \rightarrow & \mathbb{R}^3 \\ (x,y,z) & \mapsto & f((x,y,z)) = (-x+y+z, x-y+z, x+y-z). \end{array}$$

1• **Prove that f is endomorphism.**

Let $u=(x,y,z), v=(x',y',z') \in \mathbb{R}^3$, let α, $\beta \in \mathbb{R}$.
Then

$$\begin{aligned} & f(\alpha u+\beta v) \\ & = f((\alpha x+\beta x', \alpha y+\beta y', \alpha z+\beta z')) \\ & = (-(\alpha x+\beta x')+(\alpha y+\beta y')+(\alpha z+\beta z'), (\alpha x+\beta x')-(\alpha y+\beta y')+(\alpha z+\beta z'), (\alpha x+\beta x')+(\alpha y+\beta y')-(\alpha z+\beta z')) \\ & = (-\alpha x+\alpha y+\alpha z, \alpha x-\alpha y+\alpha z, \alpha x+\alpha y-\alpha z)+(-\beta x'+\beta y'+\beta z', \beta x'-\beta y'+\beta z', \beta x'+\beta y'-\beta z') \quad (01.50) \\ & = \alpha f(u)+\beta f(v). \end{aligned}$$

Hence f is endomorphism of $\mathbb{R}^3$.

2• **Define a basis of $\ker f$ and a basis of imf.**

$\ker f = \{u \in \mathbb{R}^3, f(u) = 0_{\mathbb{R}^3}\}$**, then we get**

$$\left\{ \begin{array}{r} -x+y+z=0 \\ x-y+z=0 \\ x+y-z=0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} x=0 \\ y=0 \\ z=0 \end{array} \right.$$

Hence $\ker f = \{(0,0,0)\}$ witch means $\dim \ker f = 0$,
we conclude that empty set is the basis of $\ker f$. **(0.50)**

By $\dim \mathbb{R}^3 = \dim \ker f + \dim imf$, we get that $\dim imf = 3$, hence $imf = \mathbb{R}^3$, we choose the canonical basis of $\mathbb{R}^3$ witch is $\mathcal{B} = \{(1,0,0),(0,1,0),(0,0,1)\}$. **(0.50)**

3• **Does f injective? surjective? bijective? Justify your answer.**

We have $\ker f = \{0_{\mathbb{R}^3}\}$, thus f is injective, **(0.25)**

f surjective because $imf = \mathbb{R}^3$, **(0.25)**

f is injective and surjective thus f is bijective. **(0.25)**

II) Let $\mathcal{B}' = \{u_1 = (1,1,1), u_2 = (1,0,1), u_3 = (0,0,1)\}$ be a basis of $\mathbb{R}^3$. Let M be the matrix of f with respect to the basis $\mathcal{B}'$.
1• **Prove that**

$$M = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

To define M we write $f(u_1)$, $f(u_2)$ and $f(u_3)$ in term of u_1, u_2 and u_3.

So $f(u_1) = f((1,1,1)) = (1,1,1)$.

Thus $f(u_1) = 1u_1 + 0u_2 + 0u_3$, **(0.50)**

$f(u_2) = f((1,0,1)) = (0,2,0)$.

We put $f(u_2) = \lambda 1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$,
then after small calculation we obtain $f(u_2) = 2u_1 - 2u_2 + 0u_3$. **(0.75)**

$f(u_3) = f((0,0,1)) = (1,1,-1)$,
by the same way we get $f(u_3) = 1u_1 + 0u_2 - 2u_3$. **(0.75)**

Thus

$$M = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

2• **Prove that** M **is invertible, and define** M^{-1}**.**

M is invertible because $\det M = 1(-2)(-2) = 4 \neq 0$. **(0.50)**

$$Co(M) = \begin{pmatrix} +\begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 0 & -2 \end{vmatrix} & +\begin{vmatrix} 0 & -2 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \\ +\begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 4 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix}$$ **(2.50)**

$$\Rightarrow Adj(M) = Co(M)^t = \begin{pmatrix} 4 & 4 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$ **(0.25)**

Hence $M^{-1} = \frac{1}{\det M} Adj(M) = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$.. **(0.50)**
