Answer to the second final Exam 2024

Ex01

① writing Taylor-young expansion at 0-neighborhood to 3^{rd} order.

$\sin(2x) = 2x - \frac{4}{3}x^3 + o_1(x^3)$. 01

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o_2(x^3)$ 01

② Deducing limit value of $\lim_{x\to 0} \frac{\sin(2x) - (x+1)e^x + 1}{x^2}$:

at 0-neighborhood we have

$(x+1)e^x = (x+1)\left[1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right] + (x+1)\,o_2(x^3)$

$= x + x^2 + \frac{x^3}{2} + 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o_3(x^3)$.

$(x+1)e^x = 1 + 2x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + o_3(x^3)$ 0,5

Then we thus write;

$\sin(2x) - (x+1)e^x + 1 = \cancel{2x} - \frac{4}{3}x^3 - \cancel{1} - \cancel{2x} - \frac{3x^2}{2} - \frac{2x^3}{3} + \cancel{1} + o_4(x)$

$= -\frac{5}{3}x^3 - \frac{2}{3}x^2 + o_4(x)$.

Therefore:

$\lim_{x\to 0} \frac{\sin(2x) - (x+1)e^x + 1}{x^2} = \lim_{x\to 0} \frac{-\frac{5}{3}x^3 - \frac{2}{3}x^2}{x^2}$ 0,5

$= \lim_{x\to 0}\left(-\frac{5}{3}x - \frac{2}{3}\right)$

$= -\frac{2}{3}$.

So $\lim_{x\to 0} \frac{\sin(2x) - (x+1)e^x + 1}{x^2} = -\frac{2}{3}$

③ Evaluating $\int_0^{\frac{\pi}{8}} x \sin(2x)\,dx$

we proceed by integrating by parts;

$u = x$ $\quad du = dx$

$dv = \sin(2x)\,dx$ $\quad v = \frac{-\cos(2x)}{2}$ 0,5

Then; $\int_0^{\frac{\pi}{8}} x\sin(2x)\,dx = \left[\frac{-x\cos(2x)}{2}\right]_0^{\frac{\pi}{8}} - \int_0^{\frac{\pi}{8}} \frac{-\cos(2x)}{2}\,dx$

$$= \left[\frac{-x\cos(2x)}{2}\right]_0^{\frac{\pi}{8}} + \left[\frac{\sin(2x)}{4}\right]_0^{\frac{\pi}{8}}$$

$$= -\frac{\pi}{16}\cos\left(\frac{\pi}{4}\right) + \frac{\sin\left(\frac{\pi}{4}\right)}{4}.$$

Hence $\int_0^{\frac{\pi}{8}} x\sin(2x)\,dx = \frac{\sqrt{2}}{2}\left[\frac{4-\pi}{16}\right]$ 0,5

Exo2

$$f(x) = \frac{x-1}{x^3+1}\ ;\ x \in \mathbb{R}/\{-1\}.$$

1. Determining A, B and C

we can easily find $x^3+1 = (x+1)(x^2-x+1)$.

putting $\frac{x-1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^3-x+1}$ ... (*).

- to find A we multiply (*) by $x+1$ then we let x goes to -1.

Hence $A = -\frac{2}{3}$. 0,5

- to find B we multiply (*) by x then we give the value ∞ to x.

we find $A+B=0$, wich lead us to put $B = \frac{2}{3}$ 0,75

- to find D we put $x=0$ and plug in (*) its value.

After giving A and B them values.

we find $d = -\frac{1}{3}$. 0,75

Now, we can write $f(x) = -\frac{2}{3}\frac{1}{(x+1)} + \frac{2x-1}{3(x^2-x+1)}$

② Evaluating $\int f(x)\,dx$

$\int f(x)\,dx = -\frac{2}{3}\int\frac{1}{x+1}\,dx + \frac{1}{3}\int\frac{2x-1}{x^2-x+1}\,dx$ 02,5

$$= -\frac{2}{3}\ln|x+1| + \frac{1}{3}\ln|x^2-x+1| + C / C \in \mathbb{R}.$$

③ Using Riemann sums to show $\lim\limits_{n\to+\infty} \sum\limits_{k=1}^{n} \frac{nk-n^2}{n^3+k^3} = \int_0^1 f(x)\,dx$

We Know

$$\int_0^1 f(x)\,dx = \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1-0}{n} f\left(\frac{(1-0)k}{n}\right).$$

$$= \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right).$$

$$= \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{n} \frac{\frac{k}{n}-1}{\left(\frac{k}{n}\right)^3+1}.$$

01,5

$$= \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{n} \frac{n^3\left[\frac{k}{n}-1\right]}{n^3\left[\frac{k^3}{n^3}+1\right]}$$

Therefore; $\int_0^1 f(x)\,dx = \lim\limits_{n\to+\infty} \sum\limits_{k=1}^{n} \frac{nk-n^2}{n^3+k^3}$

– Evaluating $\int_0^1 f(x)\,dx$ to find $\lim\limits_{n\to+\infty} \sum\limits_{k=1}^{n} \frac{nk-n^2}{n^3+k^3}$

$$\int_0^1 f(x)\,dx = \left[-\frac{2}{3}\ln|x+1| + \frac{1}{3}\ln|x^2-x+1|\right]_0^1$$

01

$$= -\frac{2}{3}\ln(2)$$

We Thus Conclude $\lim\limits_{n\to+\infty} \sum\limits_{k=1}^{n} \frac{nk-n^2}{n^3+k^3} = -\frac{2}{3}\ln(2)$

EX 03 :

$$y'' + 2y' + 4y = xe^x.$$

① Solving the homogenuous equation associated to (E_1)

Let (E_2) be $y'' + 2y' + 4y = 0$.

The assistant equation of (E_2) is $r^2 + 2r + 4 = 0$. 01

The discriminant is $\Delta = -12$; the $\sqrt{\Delta} = 2\sqrt{3}i$.

wich yields to $r_1 = -1 - \sqrt{3}i$ and $r_2 = -1 + \sqrt{3}i$ 01,5

we thus write $y_h = e^{-x}\left(C_1\cos(\sqrt{3}x) + C_2\sin(\sqrt{3}x)\right) / (C_1, C_2) \in \mathbb{R}^2$

② Finding y_p such that $y_p = (Ax+B)e^x$.

$$y_p = (Ax+B)e^x.$$

$$y'_p = (Ax+A+B)e^x.$$

$$y''_p = (Ax+2A+B)e^x$$

0,5

by substituting the value of y_p, y'_p, y''_p in (E_1) we find.

$$(Ax+2A+B)e^x + 2(Ax+A+B)e^x + 4(Ax+B) = xe^x.$$

Hence; $(7Ax+4A+7B)e^x = xe^x$.

by identification we find,

$$\left.\begin{array}{l} 7A = 1 \\ 4A+7B = 0. \end{array}\right\} \text{ This lead to } \left\{\begin{array}{l} A = \frac{1}{7} \\ B = -\frac{4}{49} \end{array}\right.$$

0,5

Therefore $\boxed{y_p = \left(\frac{1}{7}x - \frac{4}{49}\right)e^x}$.

3. Deducing the general solution y_g of (E_1).

we Know that $y_g = y_p + y_h$ 0,5

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Then $y_g = \left(\frac{1}{7}x - \frac{4}{49}\right)e^x + e^{-x}\left(C_1\cos(\sqrt{3}x) + C_2\sin(\sqrt{3}x)\right)$

④ Determining the unique solution s.t $h(0)=1$ and $h(1)=0$.

$h(0)=1 \Rightarrow 1 = \left(\frac{1}{7}(0) - \frac{4}{49}\right)e^0 + e^{-0}\left(C_1\cos(0) + C_2\sin(0)\right) \Rightarrow \boxed{C_1 = \frac{53}{49}}$

$h(1)=0 \Rightarrow 0 = \left(\frac{1}{7}(1) - \frac{4}{49}\right)e + e^{-1}\left(C_1\cos(\sqrt{3}) + C_2\sin(\sqrt{3})\right) \Rightarrow C_2 = \frac{53\cos\sqrt{3} + 3e^2}{49\sin(\sqrt{3})}$

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