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Duration ; 1h30min

Correction of the final exam

Solution Ex 1. (4 pts)

Answer two questions **(2pt)×2**

(1) Proving that f is a linear map.
(i) Let $X = (x_1, x_2, x_3)^t, Y = (y_1, y_2, y_3)^t \in \mathbb{R}^3$. Then, we have

$$\begin{aligned} f(X+Y) &= f(x_1+y_1, x_2+y_2, x_3+y_3) \\ &= \Big((x_1+y_1)+(x_2+y_2)+(x_3+y_3), 2(x_1+y_1)+(x_2+y_2)-(x_3+y_3)\Big) \\ &= \Big((x_1+x_2+x_3)+(y_1+y_2+y_3), (2x_1+x_2-x_3)+(2y_1+y_2-y_3)\Big) \\ &= (x_1+x_2+x_3, 2x_1+x_2-x_3)+(y_1+y_2+y_3, 2y_1+y_2-y_3) \\ &= f(X)+f(Y). \end{aligned}$$

1pt

(ii) Let $X = (x_1, x_2, x_3)^t \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$.

$$\begin{aligned} f(\lambda X) &= f(\lambda x_1, \lambda x_2, \lambda x_3) \\ &= \Big(\lambda x_1+\lambda x_2+\lambda x_3, 2\lambda x_1+\lambda x_2-\lambda x_3\Big) \\ &= \Big(\lambda(x_1+x_2+x_3), \lambda(2x_1+x_2-x_3)\Big) \\ &= \lambda(x_1+x_2+x_3, 2x_1+x_2-x_3) \\ &= \lambda f(X). \end{aligned}$$

1pt

Therefore, from (i) and (ii) we deduce that $f : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear map.

(2) Consider the set $E = \{(x, y, z) \in \mathbb{R}^3 | x+y+z = 0\}$.
(i) Clearly, E is a non-empty set, since $0_{\mathbb{R}^3} \in E$. 0.5pt
(ii) Let $X = (x_1, x_2, x_3)^t, Y = (y_1, y_2, y_3)^t \in \mathbb{R}^3$. Then we have

$$\begin{aligned} X, Y \in E &\Leftrightarrow (x_1+x_2+x_3 = 0) \text{ and } (y_1+y_2+y_3 = 0) \\ &\Rightarrow (x_1+x_2+x_3)+(y_1+y_2+y_3 = 0) \\ &\Rightarrow (x_1+y_1)+(x_2+y_2)+(x_3+y_3) = 0 \\ &\Rightarrow X+Y \in E \end{aligned}$$

1pt

(iii) Let $X = (x_1, x_2, x_3)^t \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$.

$$\begin{aligned} X \in E &\Leftrightarrow x_1+x_2+x_3 = 0 \\ &\Leftrightarrow \lambda(x_1+x_2+x_3) = 0 \\ &\Leftrightarrow \lambda x_1+\lambda x_2+\lambda x_3 = 0 \\ &\Leftrightarrow \lambda X \in E \end{aligned}$$

0.5pt

From (i),(ii) and (iii) we deduce that E is a sub-vector space of $\mathbb{R}^3$.

(3) Consider the sub-vector space $F = \{(x, y, z) \in \mathbb{R}^3 | x + 3y - 2z = 0\}$. Let $X = (x, y, z)^t \in F$, then we have $x + 3y - 2z = 0$, hence $x = -3y + 2z$. Hence

$$\begin{aligned} X = (x, y, z) &\Leftrightarrow X = (-3y + 2z, y, z) \quad \text{0.5pt} \\ &\Leftrightarrow X = (-3y, y, 0) + (2z, 0, z) \\ &\Leftrightarrow X = y(-3, 1, 0) + z(2, 0, 1), \end{aligned}$$

from which it follows that $B = \{(-3, 1, 0), (2, 0, 1)\}$ generates the sub-space F. Moreover, clearly
0.5pt the set $\{(-3, 1, 0), (2, 0, 1)\}$ is linear independent. Thus $B = \{(-3, 1, 0), (2, 0, 1)\}$ is a basis for F. The dimension of F is two $dimF = card(B) = 2$. 0.5pt 0.5pt

(4) We use the change of variable to determinate the primitive $\int \sin^3 x \cos^2 x \mathrm{d}x$. Set $t = \cos x$ so $dt = -\sin x \mathrm{d}x$. 0.25pt 0.25pt

$$\begin{aligned} \int \sin^3 x \cos^2 x \mathrm{d}x &= \int (1 - \cos^2 x) \cos^2 x \sin x \mathrm{d}x \\ &= -\int (1 - t^2) t^2 \mathrm{d}t \quad \text{0.5pt} \\ &= \int (t^4 - t^2) \mathrm{d}t \\ &= \frac{1}{5} t^5 - \frac{1}{3} t^3 + c \quad \text{0.5pt} \\ &= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + c \quad \text{0.5pt} \end{aligned}$$

(5) Set $x = \cos(t)$, so $\mathrm{d}x = -sin(t)\mathrm{d}t$, moreover if $t \to 0$ so $x \to 1$ and if $t \to \pi$ so $x \to -1$. 0.25pt 0.25pt

$$\begin{aligned} \int_0^\pi \frac{\sin(t)}{1 + \cos^2(t)} \mathrm{d}t &= -\int_1^{-1} \frac{1}{1 + x^2} \mathrm{d}x \quad \text{0.5pt} \\ &= \int_{-1}^1 \frac{1}{1 + x^2} \mathrm{d}x \\ &= \left[\arctan(x) \right\}_{x=-1}^{x=1} \quad \text{0.5pt} \\ &= \arctan(1) - \arctan(-1) \quad \text{0.25pt} \\ &= \frac{\pi}{4} - \frac{-\pi}{4} = \frac{\pi}{2} \quad \text{0.25pt} \end{aligned}$$

Solution Ex 2. (10 pts)

1) In order to show that $A^2 + A - 2I_3 = 0$ we should first calculate A^2.

$$\begin{aligned} A^2 &= A \cdot A \\ &= \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \quad \text{1pt} \end{aligned}$$

We pass to calculate $A^2 + A - 2I$

$$A^2 + A - 2I = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{0.5pt}$$

$$= \begin{pmatrix} 3-1-2 & -1+1+0 & -1+1+0 \\ -1+1+0 & 3-1-2 & -1+1+0 \\ -1+1+0 & -1+1+0 & 3-2-1 \end{pmatrix} \quad \text{0.5pt}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{0.5pt}$$

therefore, $A^2 + A - 2I = 0$.
2) We have

$$A^2 + A - 2I = 0 \Leftrightarrow A^2 + A = 2I$$
$$\Leftrightarrow A \cdot \frac{1}{2}(A+I) = I \quad \text{1pt}$$

Thus, by definition A is invertible with $A^{-1} = \frac{1}{2}(A+I)$. 0.5pt

$$A^{-1} = \frac{1}{2}(A+I) = \frac{1}{2}\left(\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad \text{1pt}$$

3) Consider the system

$$(\mathrm{S}) \quad \begin{cases} -x + y + z = 4 \\ x - y + z = 2 \\ x + y - z = 0 \end{cases}$$

(a) The system (S) can be written if matrix formulation $AX = B$ where

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

First, we calculate $det(A)$.

$$det(A) = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = -1\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = -1(0) - 1(-2) + 1(2) = 4. \quad \text{1pt}$$

Since $det(A) \neq 0$, so the system has a unique solution, given by 0.5pt

$$x = \frac{det(A_x)}{\det(A)} = \frac{\begin{vmatrix} 4 & 1 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{4}{4} = 1$$ 0.5pt

$$y = \frac{det(A_y)}{\det(A)} = \frac{\begin{vmatrix} -1 & 4 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{8}{4} = 2$$ 0.5pt

$$z = \frac{det(A_z)}{\det(A)} = \frac{\begin{vmatrix} -1 & 1 & 4 \\ 1 & -1 & 2 \\ 1 & 1 & -0 \end{vmatrix}}{\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{12}{4} = 3$$ 0.5pt

Therefore $(x, y, z) = (1, 2, 3)$.
(b) Now, we pass to solve the system (S) by using the inverse matrix method. The system (S) takes the form $AX = B$, so its solution is given by $X = A^{-1}B$. 1pt

$$X = A^{-1}B = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$ 1pt

Solution Ex 3. (6 pts)

1) Let $x \in \mathbb{R}/\{0, 1\}$.

$$\frac{a}{x} + \frac{b}{x-1} = \frac{a(x-1) + bx}{x(x-1)} = \frac{x(a+b) - a}{x(x-1)}$$ 1pt

by identification we find $a + b = 0$ and $a = -1$, so $a = -1$ and $b = 1$.
0.5pt 0.5pt
2)By the use the previous result

$$\int \frac{1}{x(x-1)} \mathrm{d}x = \int \frac{-1}{x} \mathrm{d}x + \int \frac{1}{x-1} \mathrm{d}x$$ 0.5pt
$$= -\ln|x| + \ln|x-1| + c$$ 0.5pt

3) Consider the differential equation

$$y' - y = \frac{e^x}{x(x-1)}.$$

Let y_h be solution of the homogeneous differential equation. Therefore $y_h' - y_h = 0$. This equation of the form $y' - ay = 0$ with $a = 1$ so its solution is given by $y_h = \lambda e^x$ where $\lambda \in \mathbb{R}$.
1pt

Now, we search for a particular solution y_p. Suppose that $y_p(x) = \lambda(x)e^x$, so
0.5pt

$$\begin{aligned}
y_p' - y_p = \frac{e^x}{x(x-1)} &\Rightarrow (\lambda(x)e^x)' - e^x = \frac{e^x}{x(x-1)} \\
&\Rightarrow \lambda'(x)e^x - \lambda(x)e^x + \lambda(x)e^x = \frac{e^x}{x(x-1)} \\
&\Rightarrow \lambda'(x) = \frac{1}{x(x-1)} \\
&\Rightarrow \lambda(x) = \ln|x-1| - \ln|x|.
\end{aligned}$$

1 pt

Thus, $y_p(x) = (\ln|x-1| - \ln|x|)e^x$. Consequently the general solution is given by

$$y = y_h + y_p = (\ln|x-1| - \ln|x| + \lambda)e^x, \quad \lambda \in \mathbb{R}.$$

0.5 pt