Differential Geometry (DG)-(L3-Maths)-

Correction of Final Exam

27 mai 2024

Correction of exercises

Correction of Exercice 1 –

1. 1 pt $\Rightarrow$ Let $M \subset \mathbb{R}^n$ be a 0-dimensional submanifold and $p \in M$. Through the characterization by immersion and homeorphism, there exists an open neighbourhood $U_p \subset \mathbb{R}^n$ of p such that $U_p \cap M$ is homeomorphic to an open nonempty set $\Omega \subset \mathbb{R}^0 = \{0\}$. This means that

$$\psi : \{0\} \to U_p \cap M$$

is bijective. So

$$Card\left(U_p \cap M\right) = Card\left(\{0\}\right) = 1.$$

Hence

$$U_p \cap M = \{p\}$$

and M is discrete.

1 pt $\Leftarrow$ Le $M \subset \mathbb{R}^n$ be a discrete set and $p \in M$. By the definition, there exists an open set $U_p \subset \mathbb{R}^n$ of p such that

$$U_p \cap M = \{p\}.$$

Let us consider the following diffeomorphism (constant mapping)

$$\psi : U_p \to \Omega = \{0\}$$

such that $\psi(p) = 0$. Therefore, we get

$$\psi\left(U_p \cap M\right) = \psi\left(\{p\}\right) = \{0\} = \mathbb{R}^0 = \left(\mathbb{R}^0 \times \{0\}^n\right) \cap \Omega.$$

Then, using the definition of the submanifold (coordinate charts), we obtain that $M \subset \mathbb{R}^n$ be a 0-dimensional submanifold.

2. 1 pt $\Rightarrow$ Let $M \subset \mathbb{R}^d$ be a d-dimensional submanifold and $p \in M$. Through the characterization by submersion, there exist an open neighbourhood $U_p \subset \mathbb{R}^d$ of p and a submersion g such that

$$g : U_p \to \mathbb{R}^{d-d} = \{0\}$$

and

$$g^{-1}(\{0\}) = U_p \cap M.$$

As for any $x \in U_p$, we have $g(x) = 0$, this implies

$$U_p = g^{-1}(\{0\}) = U_p \cap M.$$

Hence

$$p \in U_p \subset M.$$

Consequently, $M = \cup_{p\in M} U_p$ and so M is an open set as a union of open sets.
1 pt $\Rightarrow$ Conversely, let $p \in M$. Since M is open, there exists an open neighbourhood θ_p such that

$$p \in \theta_p \subset M \subset \mathbb{R}^d.$$

Let us consider the following diffeomorphism (Identitity mapping $I_M(x) = x$, for any $x \in M$)

$$I_M : M \to M.$$

Hence (having in mind that $\mathbb{R}^d \times \{0\}^0 = \mathbb{R}^d$)

$$I_M(\theta_p \cap M) = \theta_p = \left(\mathbb{R}^d \times \{0\}^0\right) \cap \theta_p.$$

This yields that M is a d-dimensional submanifold.

Correction of Exercice 2 –

☛ 1). Let $p = (x, y, z) \in M$ et let us consider the following $\mathscr{C}^\infty$-mapping

1 pt

$$\begin{aligned} g : \quad & U_p = \mathbb{R}^3 - \Omega \to \mathbb{R}^2 \\ & (x, y, z) \mapsto \left(x^2 + 4y^2 - 1, z - x^2 + 4y^2\right), \end{aligned}$$

where

$$\Omega = \{(0,0,z), z \in \mathbb{R}\}.$$

☛ Obviously, $p \in U_p \in \tau_{\mathbb{R}^3}$ since $0^2 + 4 \times 0^2 \neq 1$.

☛ 2 pt We have the Jacobian matrix of g is given by

$$J(g)_{(x,y,z)} = \begin{pmatrix} 2x & 8y & 0 \\ -2x & 8y & 1 \end{pmatrix}$$

and

$$\det\begin{pmatrix}2x & 0\\ -2x & 1\end{pmatrix}=2x,\quad \det\begin{pmatrix}8y & 0\\ 8y & 1\end{pmatrix}=8y.$$

The fact that $p\in U_p$ implies that $x\neq 0$ or $y\neq 0$. Therefore, g is a $\mathscr{C}^\infty$ submersion with $g^{-1}\left(\{0_{\mathbb{R}^2}\}\right)=M\cap U_p$.
Then M is a smooth $\left(\mathscr{C}^\infty\right)$ 1-dimensional submanifold of $\mathbb{R}^3$, i.e., $M(1,\infty,3)$.

☛ 1 pt 2) Let $h=(h_1,h_2,h_3)\in\mathbb{R}^3$. Using the characterization of the tangent space via a submersion, we get

$$\begin{aligned}T_{(p)}M=ker\left(D_pg\right)&=\left\{h\in\mathbb{R}^3;\ J(g)_{(x,y,z)}.h=0_{\mathbb{R}^2}\right\}\\&=\left\{(h_1,h_2,h_3)\in\mathbb{R}^3;\ xh_1=-4yh_2,\ h_3=2xh_1-8yh_2\right\}.\end{aligned}$$

☛ 2 pt 3. By routine calculation, we obtain in each case ($x\neq0$ or $y\neq0$), the following basis of $T_{(p)}M$:

$$T_{(x,y,z)}M=span\left\{(-4y,x,-16xy)\right\}.$$

Correction of Exercice 3 –

☛ 1 pts 1. Let $(a,b)\in M_1\times M_2$, which implies $a\in M_1$ and $b\in M_2$. Since M_i are d_i -submanifolds of $\mathbb{R}^{n_i}$ of class $\mathscr{C}^{k_i}$, $i=1,2$, then there exist two open neighbourhoods $U_a\subset\mathbb{R}^{n_1}$, $U_b\subset\mathbb{R}^{n_2}$, $\mathscr{C}^{k_1}$-submersion g_1 and $\mathscr{C}^{k_2}$-submersion g_2 such that

$$g_1:\quad U_a\to\mathbb{R}^{n_1-d_1}\qquad\text{with } g_1^{-1}\left(\{0_{\mathbb{R}^{n_1-d_1}}\}\right)=U_a\cap M_1$$

and

$$g_2:\quad U_b\to\mathbb{R}^{n_2-d_2}\qquad\text{with } g_2^{-1}\left(\{0_{\mathbb{R}^{n_2-d_2}}\}\right)=U_b\cap M_2.$$

1 pt

Setting

$$U_{(a,b)}:=U_a\times U_b\in\tau_{\mathbb{R}^{n_1+n_2}}$$

and

$$g:\quad U_{(a,b)}\to\mathbb{R}^{n_1-d_1}\times\mathbb{R}^{n_2-d_2}=\mathbb{R}^{n_1+n_2-(d_2+d_2)}$$

$$(x,y)\mapsto g(x,y)=\left(g_1(x),g_2(y)\right).$$

☛ Obviously, g is $\mathscr{C}^{\min(k_1,k_2)}$-mapping.

Now we show that g is a submersion. 2 pts First, we observe that

$$J(g)_{(x,y)}=\begin{pmatrix}J(g_1)_{(x)} & 0\\ 0 & J(g_2)_{(y)}\end{pmatrix}.$$

Since g_1 and g_2 are two submersions, one gets

$$rk\left(J_{g_1}\right)=n_1-d_1\quad rk\left(J_{g_2}\right)=n_2-d_2.$$

Consequently, there exist square matrices N_1 (in $J(g_1)_{(x)}$) and N_2 (in $J(g_2)_{(y)}$) such that $O(N_1) = n_1 - d_1$, $O(N_2) = n_2 - d_2$ and $\det(N_i) \neq 0$, $i = 1,2$. This yields

$$M := \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$$

is a square matrix in $J(g)_{(x,y)}$ with

$$O(M) = O(N_1) + O(N_2) = n_1 + n_2 - (d_1 + d_2)$$

and

$$\det(M) = \det(N_1) \times \det(N_2) \neq 0.$$

In view of

$$rk(J_g) = O(M) = n_1 + n_2 - (d_1 + d_2),$$

we obtain that g is a submersion.

2 pts Next, we we show that

$$g^{-1}\left(\left\{0_{\mathbb{R}^{n_1+n_2-(d_1+d_2)}}\right\}\right) \stackrel{?}{=} U_{(a,b)} \cap (M_1 \times M_2).$$

We have

$$\begin{aligned} g^{-1}\left(\left\{0_{\mathbb{R}^{n_1+n_2-(d_1+d_2)}}\right\}\right) &= g^{-1}\left(\left\{\left(0_{\mathbb{R}^{n_1-d_1}}, 0_{\mathbb{R}^{n_2-d_2}}\right)\right\}\right) \\ &= \left\{(x,y) \in U_{(a,b)}, \quad g(x,y) = \left(0_{\mathbb{R}^{n_1-d_1}}, 0_{\mathbb{R}^{n_2-d_2}}\right)\right\} \\ &= \left\{x \in U_a, \quad g_1(x) = 0_{\mathbb{R}^{n_1-d_1}} \wedge y \in U_b, \quad g_2(y) = 0_{\mathbb{R}^{n_2-d_2}}\right\} \\ &= g_1^{-1}\left(\left\{0_{\mathbb{R}^{n_1-d_1}}\right\}\right) \times g_2^{-1}\left(\left\{0_{\mathbb{R}^{n_2-d_2}}\right\}\right) \\ &= (U_a \cap M_1) \times (U_b \cap M_2) \\ &= (U_a \times U_b) \cap (M_1 \times M_2) \\ &= U_{(a,b)} \cap (M_1 \times M_2) \end{aligned}$$

and we are done. Then $M_1 \times M_2$ is a $(d_1 + d_2)$-dimensional submanifold of $\mathbb{R}^{n_1+n_2}$ of class $\mathscr{C}^{\min(k_1,k_2)}$.

0.5 pts 2. Following the question 1, the tangent space $T_{(a,b)}(M_1 \times M_2)$ is a $(d_1 + d_2)$-subspace. We have also $T_a(M_1)$ is a (d_1)-subspace and $T_a(M_2)$ is a (d_2)-subspace, which implies that

$$dim(T_a(M_1) \times T_a(M_2)) = d_1 + d_2 = dim\left(T_{(a,b)}(M_1 \times M_2)\right).$$

Hence it suffices to show that

$$T_a(M_1) \times T_a(M_2) \stackrel{?}{\subset} T_{(a,b)}(M_1 \times M_2).$$

1 pts Let $\eta, \chi > 0$. Let $h = (h_1, h_2) \in T_a(M_1) \times T_a(M_2)$, then there exist two differentiable functions α and β such that

$$\begin{aligned} \alpha : \quad &]-\eta, \eta[\to M_1 \\ & \alpha(0) = a, \quad \alpha'(0) = h_1. \end{aligned}$$

and

$$\begin{array}{ll}\beta: &]-\chi,\chi[\rightarrow M_2 \\ & \beta(0)=b, \quad \beta'(0)=h_2.\end{array}$$

2 **pts** Let observe that we can construct a differentiable function γ such that

$$\begin{array}{ll}\gamma: &]-\varepsilon,\varepsilon[\rightarrow M_1\times M_2 \\ & t\mapsto\gamma(t)=\left(\alpha(t),\beta(t)\right),\end{array}$$

where $\varepsilon=\min\left(\eta,\chi\right)>0$. Also, we have

$$\gamma(0)=\left(\alpha(0),\beta(0)\right)=(a,b)$$

and

$$\begin{array}{rcl}\gamma^{'}(0) & = & d\gamma(0)=\left(\alpha'(0),\beta'(0)\right) \\ & = & (h_1,h_2)=h.\end{array}$$

Therefore

$$h\in T_{(a,b)}(M_1\times M_2).$$

Hence

$$T_a(M_1)\times T_a(M_2)\subset T_{(a,b)}(M_1\times M_2).$$

0.5 **pts** As we have

$$dim\left(T_a(M_1)\times T_a(M_2)\right)=dim\left(T_{(a,b)}(M_1\times M_2)\right),$$

we obtain

$$T_a(M_1)\times T_a(M_2)=T_{(a,b)}(M_1\times M_2).$$